Integrability of the Bakirov system: a zero-curvature representation

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Abstract

For the Bakirov system, which is known to possess only one higher-order local generalized symmetry, a zero-curvature representation containing an essential parameter is found.

Recently, Sergyeyev [1] gave the final proof of that the Bakirov system

$$u_t = u_{xxxx} + v^2, \qquad v_t = \tfrac{1}{5} v_{xxxx} \tag{1}$$

possesses only one higher-order local generalized symmetry, namely, the sixth-order one.

In the present paper, we find a linear spectral problem associated with the Bakirov system (1), i.e. a zero-curvature representation with an essential parameter. We hope that the obtained result will be useful in future studies of the relation between Lax pairs and higher symmetries.

A zero-curvature representation (ZCR) of a system of PDEs is the compatibility condition

$$D_t X = D_x T - [X, T] \tag{2}$$

of the over-determined linear problem

$$\Psi_x = X\Psi, \qquad \Psi_t = T\Psi, \tag{3}$$

where D_t and D_x are the total derivatives, X and T are $n \times n$ matrix functions of independent and dependent variables and finite-order derivatives of

dependent variables, the square brackets denote the matrix commutator, Ψ is a column of n functions of independent variables, and the ZCR (2) is satisfied by any solution of the represented system of PDEs. Two ZCRs are equivalent if they are related by a gauge transformation

$$\begin{aligned} X' &= GXG^{-1} + (D_x G)\, G^{-1}, \\ T' &= GTG^{-1} + (D_t G)\, G^{-1}, \\ \Psi' &= G\Psi, \qquad \det G \neq 0, \end{aligned} \tag{4}$$

where G is an $n \times n$ matrix function of independent and dependent variables and finite-order derivatives of dependent variables.

Our aim is to find a ZCR (2) of the Bakirov system (1). Assuming for simplicity that $X = X(u, v)$ and $T = T(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx})$, and using (1), we rewrite (2) in the equivalent form

$$\left(u_{xxxx} + v^2\right) \frac{\partial X}{\partial u} + \frac{1}{5} v_{xxxx} \frac{\partial X}{\partial v} - D_x T + [X, T] = 0. \tag{5}$$

Since (5) cannot be a system of ODEs restricting solutions of (1), it must be an identity with respect to u and v, and therefore u, v and all the derivatives of u and v are mutually independent variables in (5). This allows us to solve (5) and obtain the following expressions for the matrices X and T:

$$\begin{aligned} X &= Pu + Qv + R, \\ T &= Pu_{xxx} + \tfrac{1}{5} Q v_{xxx} + [R, P]\, u_{xx} + \tfrac{1}{5} [R, Q]\, v_{xx} + [R, [R, P]]\, u_x \\ &\quad + \tfrac{1}{5} [R, [R, Q]]\, v_x + [R, [R, [R, P]]]\, u + \tfrac{1}{5} [R, [R, [R, Q]]]\, v + S, \end{aligned} \tag{6}$$

where P, Q, R and S are any $n \times n$ constant matrices satisfying the following commutator relations:

$$\begin{aligned} & P = -\tfrac{1}{5} [Q, [R, [R, [R, Q]]]], \qquad [P, Q] = 0, \\ & [P, [Q, R]] = 0, \qquad [P, [R, P]] = 0, \qquad [Q, [R, Q]] = 0, \\ & [[R, P], [R, Q]] = 0, \qquad [P, [R, [R, [R, P]]]] = 0, \\ & [[R, P], [R, [R, Q]]] = 0, \qquad [P, S] + [R, [R, [R, [R, P]]]] = 0, \\ & [Q, S] + \tfrac{1}{5} [R, [R, [R, [R, Q]]]] = 0, \qquad [R, S] = 0. \end{aligned} \tag{7}$$

We have to find a solution of (7), nontrivial in the following sense: X contains both u and v, i.e. (2) gives expressions for both equations of (1); $[X, T] \neq 0$, because commutative ZCRs are simply matrices of conservation laws (for this reason, and without loss of generality, the matrices P, Q, R and S are set to be traceless); X contains a parameter ('essential' or 'spectral')

which cannot be removed by gauge transformations (4). We solve (7), using the *Mathematica* computer algebra system [2], successively taking Q in all possible Jordan forms, suppressing the excessive arbitrariness of solutions by transformations (4) with constant G, and increasing the matrix dimension n. The cases of 2×2 and 3×3 matrices contain no nontrivial solutions of (7), while the case 4×4 gives us the following:

$$P = \begin{pmatrix} 0 & 0 & \frac{8}{5}z\left(-3+6z-11z^2\right)\alpha^3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

$$R = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & z\alpha & 0 & \alpha \\ 0 & 0 & (-1+2z)\,\alpha & 0 \\ 0 & \left(-3+6z-11z^2\right)\alpha & 0 & -3z\alpha \end{pmatrix}, \tag{10}$$

$$S = \begin{pmatrix} S_{11} & 0 & 0 & 0 \\ 0 & S_{22} & 0 & S_{24} \\ 0 & 0 & S_{33} & 0 \\ 0 & S_{42} & 0 & S_{44} \end{pmatrix} \tag{11}$$

with

$$\begin{aligned} S_{11} &= \tfrac{8}{5}\left(2-12z+21z^2-18z^3+3z^4\right)\alpha^4, \\ S_{22} &= \tfrac{8}{5}\left(3-10z+15z^2-4z^4\right)\alpha^4, \\ S_{24} &= \tfrac{8}{5}\left(-1+3z+z^2-3z^3\right)\alpha^4, \\ S_{33} &= \tfrac{8}{5}\left(-8+28z-39z^2+22z^3-7z^4\right)\alpha^4, \\ S_{42} &= \tfrac{8}{5}\left(3-15z+26z^2-18z^3-29z^4+33z^5\right)\alpha^4, \\ S_{44} &= \tfrac{8}{5}\left(3-6z+3z^2-4z^3+8z^4\right)\alpha^4, \end{aligned} \tag{12}$$

where α is an arbitrary parameter, and z is any of the four roots

$$\begin{aligned} z_{1,2} &= \frac{1}{20}\left(9+\mathrm{i}\sqrt{39} \pm \sqrt{-138-2\mathrm{i}\sqrt{39}}\right), \\ z_{3,4} &= \frac{1}{20}\left(9-\mathrm{i}\sqrt{39} \pm \sqrt{-138+2\mathrm{i}\sqrt{39}}\right) \end{aligned} \tag{13}$$

of the algebraic equation

$$3 - 12z + 21z^2 - 18z^3 + 10z^4 = 0. \tag{14}$$

Consequently, a nontrivial ZCR (2) of the Bakirov system (1) is determined by the following 4×4 matrices X and T:

$$X = \begin{pmatrix} \alpha & v & \frac{8}{5}z\left(-3+6z-11z^2\right)\alpha^3 u & 0 \\ 0 & z\alpha & v & \alpha \\ 0 & 0 & (-1+2z)\,\alpha & 0 \\ 0 & \left(-3+6z-11z^2\right)\alpha & 0 & -3z\alpha \end{pmatrix} \tag{15}$$

and

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ 0 & T_{22} & T_{23} & T_{24} \\ 0 & 0 & T_{33} & 0 \\ 0 & T_{42} & T_{43} & T_{44} \end{pmatrix} \tag{16}$$

with

$$\begin{aligned}
T_{11} &= \tfrac{8}{5}\left(2 - 12z + 21z^2 - 18z^3 + 3z^4\right)\alpha^4, \\
T_{12} &= \tfrac{1}{5}\left(-4\left(2 - 3z + 6z^2 + 3z^3\right)\alpha^3 v - 2\left(1 - 2z + 5z^2\right)\alpha^2 v_x \right. \\
&\quad \left. + (1-z)\,\alpha v_{xx} + v_{xxx}\right), \\
T_{13} &= \tfrac{8}{5}z\left(-3 + 6z - 11z^2\right)\alpha^3\left(8(1-z)^3\alpha^3 u + 4(1-z)^2\alpha^2 u_x \right. \\
&\quad \left. +2(1-z)\,\alpha u_{xx} + u_{xxx}\right), \\
T_{14} &= \tfrac{1}{5}\alpha\left(4z(-3+z)\,\alpha^2 v - 2(1+z)\,\alpha v_x - v_{xx}\right), \\
T_{22} &= \tfrac{8}{5}\left(3 - 10z + 15z^2 - 4z^4\right)\alpha^4, \\
T_{23} &= \tfrac{1}{5}\left(4\left(-2 + 9z - 18z^2 + 19z^3\right)\alpha^3 v - 2\left(1 - 2z + 5z^2\right)\alpha^2 v_x \right. \\
&\quad \left. + (1-z)\,\alpha v_{xx} + v_{xxx}\right), \\
T_{24} &= \tfrac{8}{5}\left(-1 + 3z + z^2 - 3z^3\right)\alpha^4, \\
T_{33} &= \tfrac{8}{5}\left(-8 + 28z - 39z^2 + 22z^3 - 7z^4\right)\alpha^4, \\
T_{42} &= \tfrac{8}{5}\left(3 - 15z + 26z^2 - 18z^3 - 29z^4 + 33z^5\right)\alpha^4, \\
T_{43} &= \tfrac{1}{5}\left(-3 + 6z - 11z^2\right)\alpha\left(4z(-3+5z)\,\alpha^2 v \right. \\
&\quad \left. + (2-6z)\,\alpha v_x + v_{xx}\right), \\
T_{44} &= \tfrac{8}{5}\left(3 - 6z + 3z^2 - 4z^3 + 8z^4\right)\alpha^4.
\end{aligned} \tag{17}$$

Let us remind that, in (15)–(17), z is any of the roots (13) of the equation (14), and α is an arbitrary parameter.

Now, we have to prove that α is an essential parameter, i.e. that α cannot be removed from the obtained ZCR by a gauge transformation (4). We do this, using the method of gauge-invariant description of ZCRs of evolution equations [3] (see also the independent work [4], based on the very general and abstract study of ZCRs [5]). Since the matrix X (15) does not contain derivatives of u and v, the two characteristic matrices of the obtained ZCR are simply $C_u = \partial X/\partial u = P$ and $C_v = \partial X/\partial v = Q$. We take one of them, $C_u = P$ (8), introduce the operator ∇_x, defined as $\nabla_x M = D_x M - [X, M]$ for any 4×4 matrix function M, compute $\nabla_x C_u$, and find that

$$\nabla_x C_u + 2\,(1 - z)\,\alpha C_u = 0. \tag{18}$$

In the terminology of [3], the relation (18) is one of the two closure equations of the cyclic basis. The scalar coefficient $2\,(1 - z)\,\alpha$ in (18) is an invariant with respect to the gauge transformations (4), because the matrices C_u and $\nabla_x C_u$ are transformed as $C'_u = G C_u G^{-1}$ and $\nabla'_x C'_u = G\,(\nabla_x C_u)\,G^{-1}$ (see [3] and [4]). The explicit dependence of the invariant $2\,(1 - z)\,\alpha$ on the parameter α shows that this parameter cannot be 'gauged out' from the matrix X (15).

We think that the ZCR of the Bakirov system, found in this paper, can be used in future studies of the relation between Lax pairs, recursion operators and generalized symmetries. The following problems arise from the obtained result. Is it possible to derive a recursion operator for the Bakirov system from its ZCR? If not, why? If there exists a recursion operator of the Bakirov system, why the operator does not produce infinitely many local generalized symmetries?

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References

[1] A. Sergyeyev, Symmetries and integrability: Bakirov system revisited, *J. Phys. A: Math. Gen.* **34** (2001) 4983–4990.

[2] S. Wolfram, *The Mathematica Book*, 4th ed. (Cambridge University Press, Cambridge, 1999).

[3] S.Yu. Sakovich, On zero-curvature representations of evolution equations, *J. Phys. A: Math. Gen.* **28** (1995) 2861–2869.

[4] M. Marvan, A direct procedure to compute zero-curvature representations. The case sl_2, *International Conference on Secondary Calculus and Cohomological Physics*, Moscow, Russia, August 24–31, 1997, 9 pp. (ELibEMS, www.emis.de/proceedings/SCCP97).

[5] M. Marvan, On zero-curvature representations of partial differential equations, *Differential Geometry and Its Applications*, eds. O. Kowalski and D. Krupka (Silesian University, Opava, 1993) 103–122. (ELibEMS, www.emis.de/proceedings/5ICDGA).